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ON DOMAINS WHICH HAVE PRIME IDEALS THAT ARE LINEARLY ORDERED

Ayman Badawi
Department of Mathematics
Emory & Henry College
Emory, VA 24327

Introduction. Throughout this paper the letter R denotes a commutative integral domain with identity and quotient field K. If I is an ideal of a ring A, then Rad(I) denotes the radical of I. A domain R is a valuation domain if and only if for every $a,b \in R$, either $a|b$ or $b|a$. Recall that an integral domain R is a GCD domain if any two elements in R have a greatest common divisor. It is well-known that a GCD domain R which has prime ideals that are linearly ordered is a valuation domain. The purpose of this paper is to provide an alternative proof of this fact. Furthermore, we will give a characterization of divided domains and another characterization of pseudo-valuation domains that are somewhat analogous to the characterization of valuation domains given above.

We start by recalling the following definitions

Definition 1. A domain R is called a divided domain in the sense of [6] if every prime ideal of R is comparable to every principal ideal of R.

Definition 2. A prime ideal P of R is called strongly prime in the sense of [7] if whenever $x,y \in K$

and $xy \in P$, then $x \in P$ or $y \in P$. If every prime ideal of R is strongly prime, then R is called a pseudo-valuation domain [abbreviated PVD].

We start with the following Theorem :

Theorem 1. The following statements are equivalent for a commutative ring A with identity.

(1) The prime ideals of A are linearly ordered.

(2) The radical ideals of A are linearly ordered.

(3) Each proper radical ideal of A is prime.

(4) The radical ideals of principal ideals of A are linearly orderd.

(5) For each $a,b \in A$, there is an $n \geq 1$ such that either $a|b^n$ or $b|a^n$.

Proof. (1) ⇒ (2). Let I be a proper ideal of A and P be the minimum prime ideal of A over I. Then $Rad(I)=P$. (2) ⇒ (1). This requires no comment. (2) ⇒ (3). Let I be a proper ideal of A and P be the minimum prime ideal of A over I. Then $Rad(I) = P$. (3) ⇒ (1). Suppose that P,Q are two distinct prime ideals of A. Let $I = P \bigcap Q$. Then $Rad(I) = I$ is a prime ideal of A. But this is possible only if $P \subset Q$ or $Q \subset P$. (2) ⇒ (4). Clear. (4) ⇒ (5). This is clear by the definition of radical ideals. (5) ⇒ (1). Suppose that P, Q are two distinct prime ideals of A. Now, suppose that there is a $p \in P - Q$. Then for every $q \in Q$ there is an $n \geq 1$ such that $p|q^n$. Therefore $q \in P$. ■

In view of the above Theorem, we have :

Corollary 1. Suppose the prime ideals of a commutative ring A with identity are linearly ordered and a,b are nonzero nonunit elements of A. Let P be the minimum prime ideal of A that contains a and Q be the minimum prime ideal of A that contains b.

Then $P=Q$ if and only if there exist $n\geq 1$, $m\geq 1$ such that $a|b^n$ and $b|a^m$.

Proof. This is just the observation that $P = \mathrm{Rad}((a)) = \mathrm{Rad}((b)) = Q$. ■

The following result appeared in [9, Theorem 1], [5, Corollary 4.3], [10, Corollary 3.8], and [11, Proposition A]. In view of the above Theorem, we give a different proof of it.

Proposition 1. A GCD domain R which has prime ideals that are linearly ordered is a valuation domain.

Proof. Let a,b be nonzero nonunit elements of R, and let $f=\gcd(a,b)$. Suppose that f is associated in R to neither a nor b. Let $d=a/f$, and $g=b/f$. Then neither d nor g is a unit of R. Thus, by Theorem 1 there exists $m\geq 1$ such that either $d|g^m$ or $g|d^m$. But it is well-known that $\gcd(d,g)=1$ and therefore for every $n\geq 1$ $\gcd(d,g^n) = \gcd(g,d^n) = 1$ (see [8, Theorem 49] .) Hence, d or g is a unit of R, which is a contradiction. Thus, the assumption that f is associated in R to neither a nor b is invalid. Hence, $a|b$ or $b|a$. Therefore, R is a valuation domain. ■

The following Proposition gives a characterization of divided domains in the sense of [6].

Proposition 2. The following statements are equivalent for an integral domain R.

(1) R is a divided domain.

(2) For every pair of proper ideals I, J of R, the ideals I and Rad(J) are comparable.

(3) For every $a,b\in R$, the ideals (a) and Rad((b)) are comparable.

(4) For every $a,b\in R$, either $a|b$ or $b|a^n$ for some $n\geq 1$.

Proof. (1) ⇒ (2). Suppose R is a divided domain. Then by the definition of divided domains, the prime ideals of R are linearly ordered. Let I, J be two proper ideals of R. Since R is divided, Rad(J) = P is prime by Theorem 1 above. Thus, either $(a) \subset P$ or $P \subset (a)$ for every $a \in I$ since R is divided. Hence, the ideals I, Rad(J) are comparable. (2) ⇒ (3). This requires no comment. (3) ⇒ (4). Clear. (4) ⇒ (1). Suppose that for every $a, b \in R$, either $a|b^n$ for some $n \geq 1$ or $b|a$. Let P be a prime ideal of R and $s \in R-P$ and $p \in P$. Since for every $n \geq 1$ p does not divide s^n, $s|p$. Hence, P is comparable to every principal ideal of R. Therefore R is a divided domain. ■

Let R be a PVD, and a,b be nonzero nonunit elements of R. Suppose that a does not divide b and b does not divide a^2. Then $c=b/a$ and $g=a^2/b$ are elements in K-R. Let M be a maximal ideal of R that contains a. Then $cg = a \in M$. But neither c nor g is an element of M. A contradiction, since M is strongly prime. Thus, $a|b$ or $b|a^2$. Hence, by Theorem 1 the prime ideals of R are linearly ordered. In particular, R is quasilocal. This argument provides an alternative proof of [7, Corollary 1.3].

Anderson [1, Proposition 3.1] proved that a quasi-local domain R with maximal ideal M is a PVD if and only if for every $x \in K$, either $xR \subset M$ or $M \subset xR$, that is, if for every $a,b \in R$, either $aM \subset bR$ or $bR \subset aM$. In view of [1, Proposition 3.1] and Theorem 1, we now give several other Characterizations of pseudo-valuation domains.

Proposition 3. Let N be the set of all nonunit elements of an integral domain R. The following statements are equivalent.

(1) R is a PVD with the maximal ideal N.

(2) For each pair I, J of ideals of R, either $J \subset I$ or $IB \subset J$ for every proper ideal B of R.

(3) For every $a, b \in R$, either $bR \subset aR$ or $acR \subset bR$ for every nonunit $c \in R$.

(4) For every $a, b \in R$, either $a|b$ or $b|ac$ for every nonunit $c \in R$.

(5) For every $a, b \in R$, either $bR \subset aR$ or $aN \subset bR$.

(6) For every $a, b \in R$, either $bN \subset aR$ or $aR \subset bN$.

Proof. (1) $\Rightarrow$ (2). Let I, J be ideals of R and B be a proper ideal of R. Suppose that J is not a subset of I and BI is not a subset of J. Then there exist $j \in J - I$ and $ib \in IB$ for some $i \in I$ and $b \in B$ such that $j/i \in K - R$ and $ib/j \in K - R$. But $(j/i)(bi/j) = b \in N$ and neither $j/i \in N$ nor $ib/j \in N$, which is a contradiction. (2) $\Rightarrow$ (3). Clear. (3) $\Rightarrow$ (4). Clear. (4) $\Rightarrow$ (1). Suppose that for every $a, b \in R$ and any nonunit c in R, either $a|b$ or $b|ac$. Let a be any nonunit element of R and $b \in R$. Then either $a|b$ or $b|a^2$. Hence, the prime ideals of R are linearly ordered by Theorem 1. In particular, R is quasilocal with maximal ideal N. By [2, Proposition 4.8] (see also [4, Proposition 2]), it suffices to show that N is strongly prime. Suppose that $xy \in N$ for some $x, y \in K$. If $x \in R$ or $y \in R$, then it is easy to see that $x \in N$ or $y \in N$. Hence, suppose that $x, y \in K - R$. Write $x = b/a$ and $y = c/d$ for some $a, b, c, d \in R$. Since $x = b/a \in K - R$ and $xy = bc/ad \in N$, $b|a(bc/ad)$. Thus, $y = c/d \in R$, which is a contradiction. Therefore, if $xy \in N$ for some $x, y \in K$, then $x \in N$ or $y \in N$. (4) $\Longleftrightarrow$ (5). Clear. (6) $\Rightarrow$ (1). Let $a, b \in R$. Then either $a|b$ or $b|a^2$.

Hence, the prime ideals of R are linearly ordered. In particular, R is quasilocal with the maximal ideal N. Hence, R is a PVD by [1, Proposition 3.1]. (1) ⇒ (6). Again, This is just a restatement of [1, Proposition 3.1]. ■

An immediate consequence of the above Proposition is [7, Proposition 1.1]. We state it here as a corollary.

Corollary 2. Every valuation domain is a PVD.

RELATED RESULTS

Throughout this section the letter N denotes the set of all nonunit elements of R, and G denotes the group of divisibility of R. If I is an ideal of R, then $I : I = \{ x \in K : xI \subset I \}$.

Anderson [1, Proposition 3.10] proved the following result.

Fact 1 [1, Proposition 3.10]. The following statements are equivalent for a quasilocal domain R with maximal ideal M.

(1) For every $a,b \in R$, either $aM \subset bR$ or $bM \subset aR$.

(2) For every $a,b \in R$, either $aM \subset bM$ or $bM \subset aM$.

In view of Fact 1 above and Theorem 1, we have the following result.

Proposition 4. The following statements are equivalent for an integral domain R. Furthermore, if R satisfies any of the following conditions, then R is quasilocal with the maximal ideal N and $N : N$ is a valuation domain.

(1) For every $a,b \in R$, either $aN \subset bR$ or $bN \subset aR$.

(2) For every $a,b \in R$, either $aN \subset bN$ or $aN \subset bN$.

Proof. Suppose that R satisfies (1) or (2) above. Let $a,b \in R$. Then $a|b^2$ or $b|a^2$. Hence, the prime ideals of R are linearly ordered by Theorem 1. In particular, R is quasilocal with the maximal ideal N. Thus, (1) and (2) are equivalent by Fact 1 above. Now, if R satisfies (1) or (2), then $N : N$ is a valuation domain by [1, Corollary 3.4]. ∎

In light of Proposition 4 above, we have the following result.

Proposition 5. The following statements are equivalent for an integral domain R.

(1) R is quasilocal with the maximal ideal N such that $N : N$ is a valuation domain.

(2) For every $a,b \in R$, either $aN \subset bR$ or $bN \subset aR$.

(3) For every $a,b \in R$, $a|bc$ for every nonunit $c \in R$ or $b|ac$ for every nonunit $c \in R$.

(4) For every $a,b \in R$, either $aN \subset bN$ or $bN \subset aN$.

Proof. Clearly, (3) is a restatement of (2). By Proposition 4 above, we now only need show that (1) $\Rightarrow$ (2). Hence, suppose that R is quasilocal with the maximal ideal N and $N : N$ is a valuation domain. For nonzero $a,b \in R$, either $a/bN \subset N$ or $b/aN \subset N$ since $N : N$ is a valuation domain. Thus, $aN \subset bR$ or $bN \subset aR$. ∎

Remark 1. It is well-known that a quasilocal domain R with maximal ideal M is a PVD iff $M : M$ is a valuation domain with maximal ideal M (see [3, Proposition 2.5].) So it is natural to ask whether the condition $aN \subset bR$ or $bN \subset aR$ implies that the domain in the above Proposition is a PVD. The answer is negative and for a counter-example see [1, Example 3.2].

Combining [1, Proposition 3.10, Corollary 3.4, Corollary 3.8, Proposition 3.11 (b), Proposition 3.12, Proposition 4.3, and Proposition 5.2] with Proposition 5, we arrive at the following Corollary.

Corollary 3. The following statements are equivalent for an integral domain R.

(1) R is quasilocal with maximal ideal M such that $M : M$ is a valuation domain.

(2) For each nonzero prime ideal P of R, $P : P$ is a valuation domain.

(3) The prime ideals of R are linearly ordered and if M is the maximal ideal of R, then $M : M$ is a valuation domain.

(4) For every $a,b \in R$, either $aN \subset bR$ or $bN \subset aR$.

(5) For every $a,b \in R$, either $aN \subset bN$ or $bN \subset aN$.

(6) For every $a,b \in R$, either $a|bc$ for every nonunit $c \in R$ or $b|ac$ for every nonunit $c \in R$.

(7) For each $g \in G$, either $g > h$ for all $h \in G$ with $h < 0$ or $g < h$ for all $h \in G$ with $h > 0$.

(8) There is a valuation overring V of R and a maximal ideal J of R which is also an ideal of V.

(10) For each $x \in K$ and maximal ideal M of R, xM and M are comparable.

(11) R is quasilocal with maximal ideal M such that for every $a,b \in R$, either $aM \subset bR$ or $bM \subset aR$.

(12) R is quasilocal with maximal ideal M such that for every $a,b \in R$, either $aM \subset bM$ or $bM \subset aM$.

(13) For some maximal ideal M of R, xM and M are comparable for each $x \in K$.

(14) For each $x \in K$, there is a maximal ideal M of R so that xM and M are comparable.

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